Nonlinear Schrödinger equation on metric graphs

COMPLEX Doctoral School

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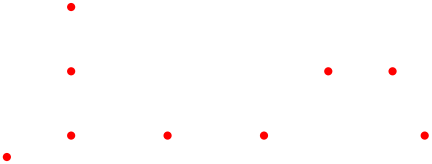
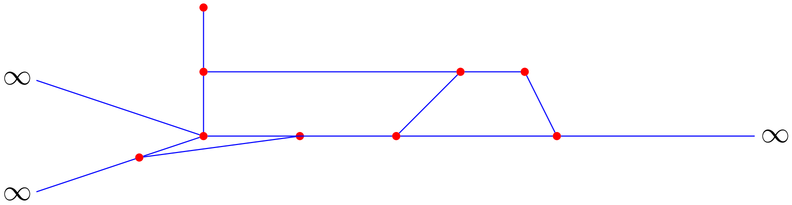
What is a metric graph?

A metric graph is made of vertices



What is a metric graph?

A metric graph is made of vertices and of edges joining the vertices or going to infinity.



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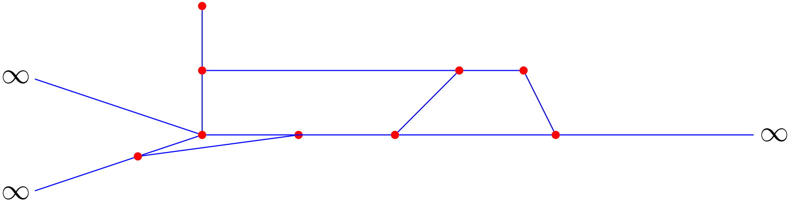

- *metric* graphs: the length of edges are important.

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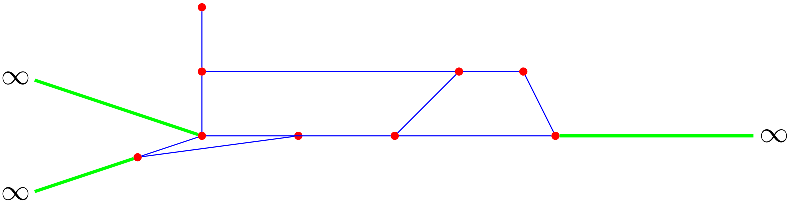
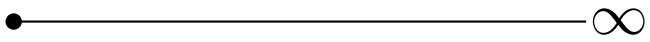


- *metric* graphs: the length of edges are important.
- the edges going to infinity are halflines and have *infinite length*.

Constructions based on halflines



The halfline

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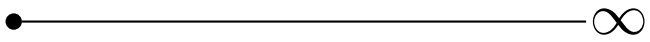

The halfline

The line

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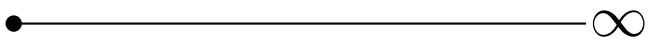

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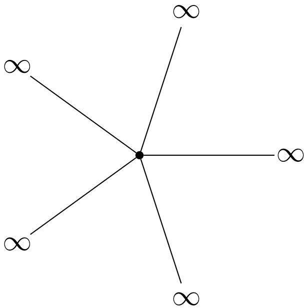

The 5-star graph

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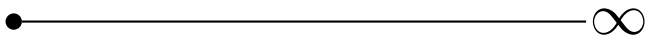

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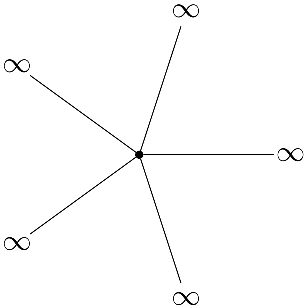

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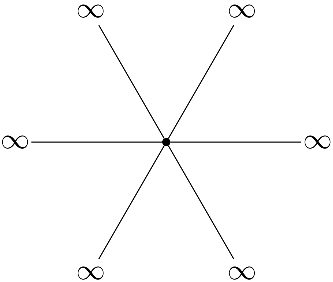
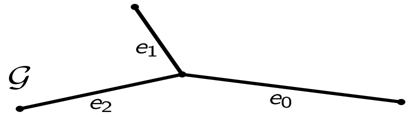


The 6-star graph

Functions defined on metric graphs



A metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) et e_2 (length 3)

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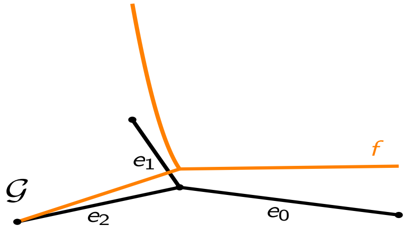

A metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) et e_2 (length 3), a function $f : \mathcal{G} \to \mathbb{R}$

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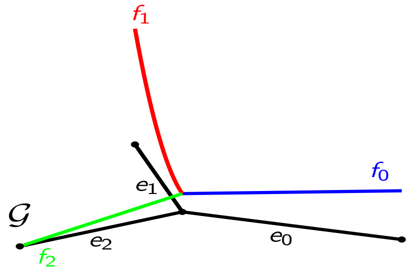

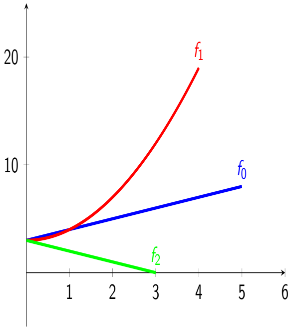

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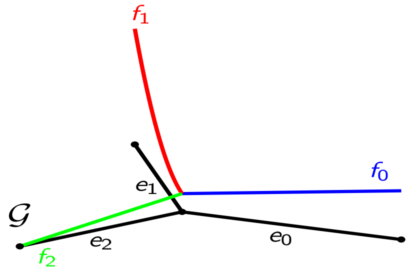

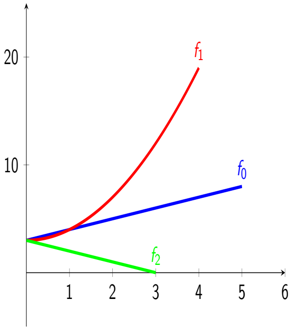

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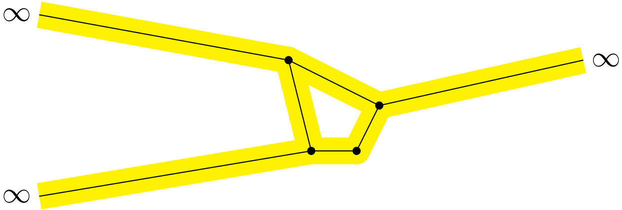
$$\int_{\mathcal{G}} f \, \mathrm{d}x \stackrel{\text{def}}{=} \int_0^5 f_0(x) \, \mathrm{d}x + \int_0^4 f_1(x) \, \mathrm{d}x + \int_0^3 f_2(x) \, \mathrm{d}x$$

Why studying metric graphs?

Modeling structures where *only one spatial direction is important.*



A « fat graph » and the underlying metric graph

An application: atomtronics

- A *boson*[1] is a particle with integer spin.

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- This is really remarkable: *macroscopic quantum phenomenon!*
- Since 2000: emergence of *atomtronics*, which studies circuits guiding the propagation of ultracold atoms.

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$$H^1_\mu(\mathcal{G}) = \left\{ u : \mathcal{G} \to \mathbb{R} \;\middle|\; u \text{ is continuous}, u, u' \in L^2(\mathcal{G}), \int_{\mathcal{G}} |u|^2 = \mu \right\}$$

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and we consider the energy minimization problem

$$\inf_{u \in H^1_\mu(\mathcal{G})} \frac{1}{2} \int_{\mathcal{G}} |u'|^2 - \frac{1}{p} \int_{\mathcal{G}} |u|^p,$$

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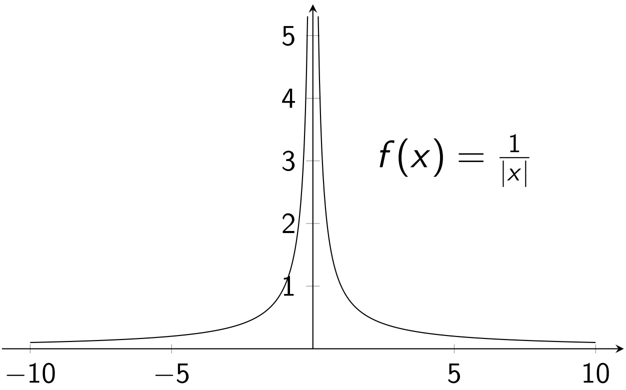
where $2 < p < 6$ (Bose-Einstein: $p = 4$).

Infimum vs minimum



Then

$$\inf_{\mathbb{R}} f = 0$$

but the infimum is not attained (i.e. is not a minimum).

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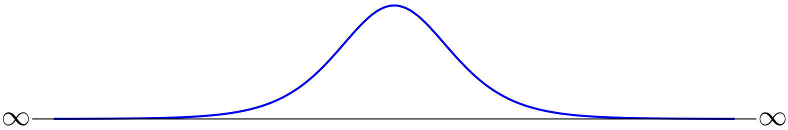
The real line: $\mathcal{G} = \mathbb{R}$



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where the *soliton* φ_μ is the unique strictly positive, even, and of mass μ solution to an equation of the form

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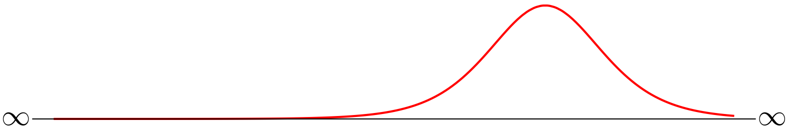

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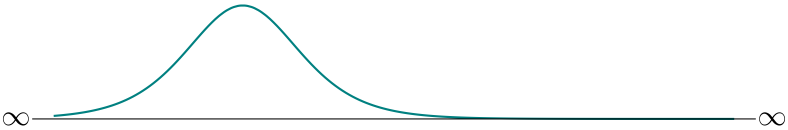

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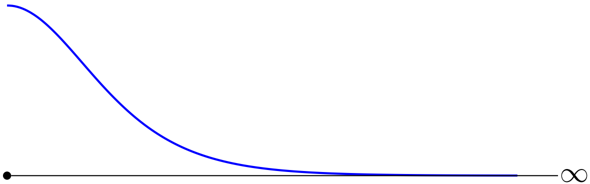
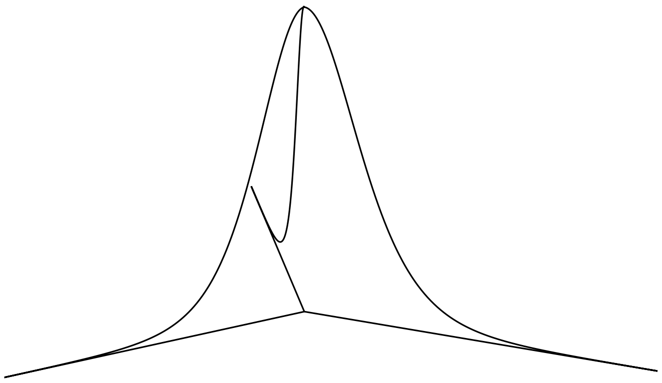
The halfline: $\mathcal{G} = \mathbb{R}^+ = [0, +\infty[$



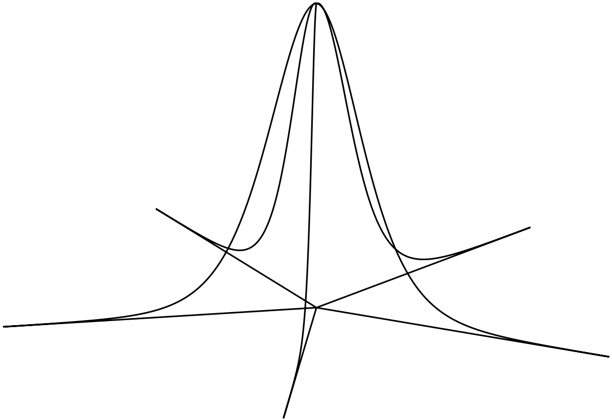
$$\mathcal{S}_\mu(\mathbb{R}^+) = \left\{ \pm\varphi_{2\mu}(x)_{|\mathbb{R}^+} \right\}$$

Solutions are *half-solitons*: no more translations!

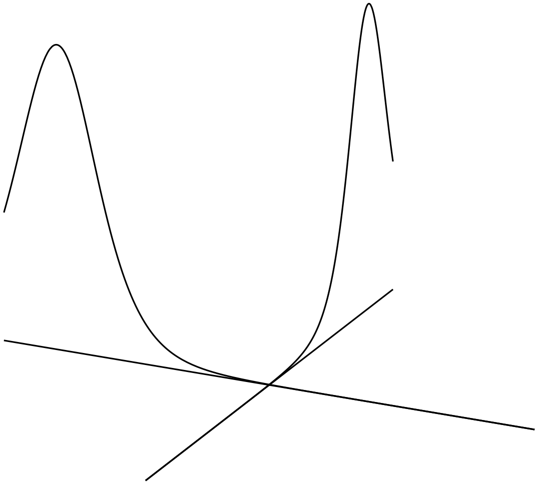
The positive solution on the 3-star graph



The positive solution on the 5-star graph



A continuous family of solutions on the 4-star graph



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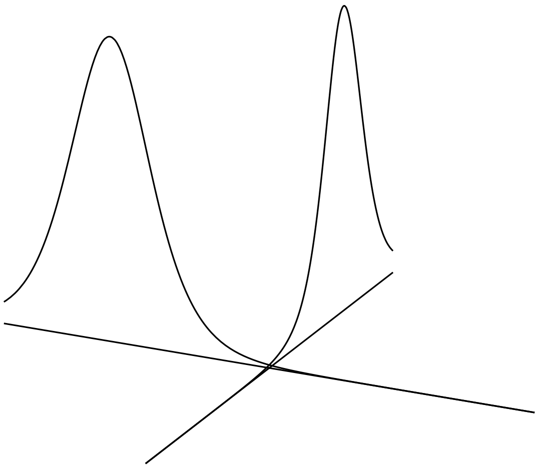

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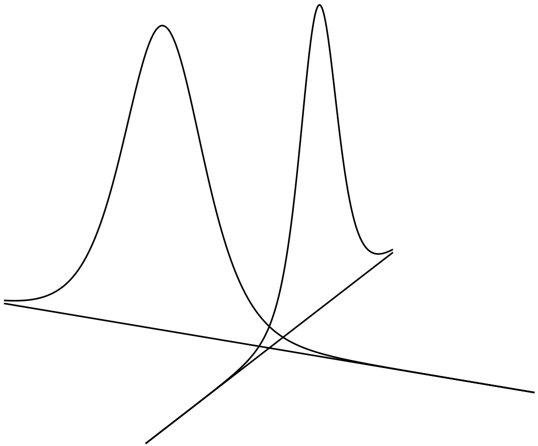

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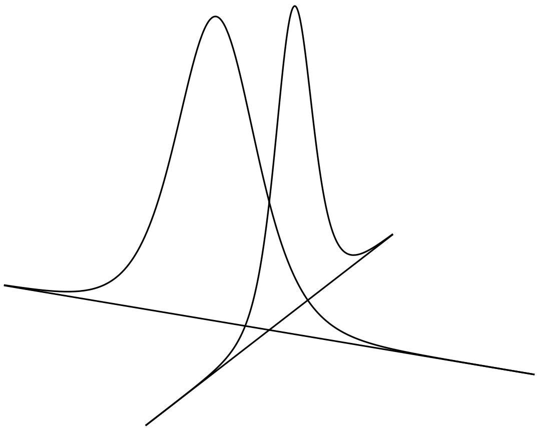

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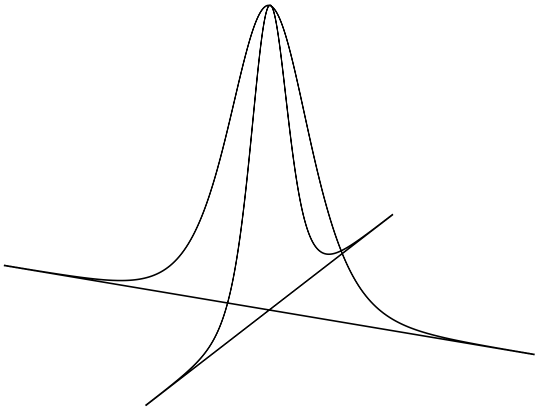

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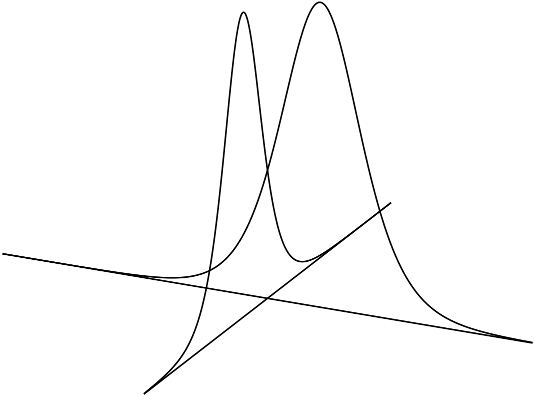

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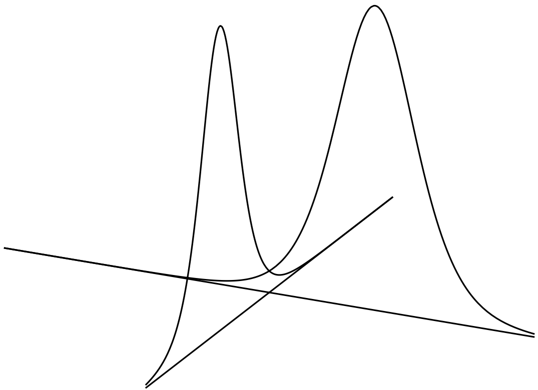

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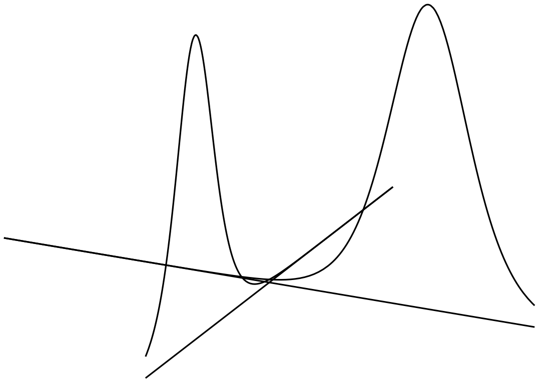

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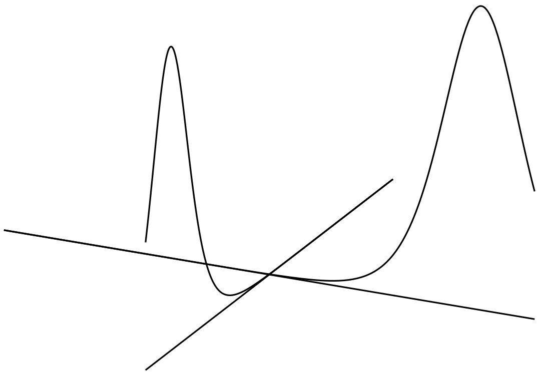

Two energy levels

- The « ground state » energy level is given by

$$c_\mu(\mathcal{G}) = \inf_{u \in H^1_\mu(\mathcal{G})} \frac{1}{2}\int_{\mathcal{G}} |u'|^2 - \frac{1}{p}\int_{\mathcal{G}} |u|^p.$$

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- A *ground state* is a function $u \in H^1_\mu(\mathcal{G})$ with level $c_\mu(\mathcal{G})$. It is a solution of the differential system (NLS).
- We can also consider the minimal level attained by the solutions of (NLS):

$$\sigma_\mu(\mathcal{G}) = \inf_{u \in \mathcal{S}_\mu(\mathcal{G})} \frac{1}{2}\int_{\mathcal{G}} |u'|^2 - \frac{1}{p}\int_{\mathcal{G}} |u|^p.$$

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- A *minimal action solution* of the problem is a solution $u \in H^1_\mu(\mathcal{G})$ of the differential problem (NLS) of level $\sigma_\mu(\mathcal{G})$.

An example: star graphs

The level of the mass μ soliton on the real line is given by

$$s_\mu = \frac{1}{2}\int_{\mathcal{G}} |\varphi_\mu'|^2 - \frac{1}{p}\int_{\mathcal{G}} |\varphi_\mu|^p.$$

For a N-star graph with $N \geq 3$, we have

$$s_\mu = c_\mu(\mathcal{G}) < \sigma_\mu(\mathcal{G}) = \frac{N}{2} s_\mu.$$

Four cases

An analysis shows that four cases are possible:

A1) $c_\mu(\mathcal{G}) = \sigma_\mu(\mathcal{G})$ and both infima are attained;

A2) $c_\mu(\mathcal{G}) = \sigma_\mu(\mathcal{G})$ and neither infima is attained;

B1) $c_\mu(\mathcal{G}) < \sigma_\mu(\mathcal{G})$, $\sigma_\mu(\mathcal{G})$ is attained but not $c_\mu(\mathcal{G})$;

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Question

Are those four cases really possible for metric graphs?

Answer to the question

Theorem (De Coster, Dovetta, G., Serra (to appear))

For every $p \in]2,6[$, every $\mu > 0$, and every choice of alternative between A1, A2, B1, B2, there exists a metric graph $\mathcal{G}$ where this alternative occurs.

Thanks for your attention!

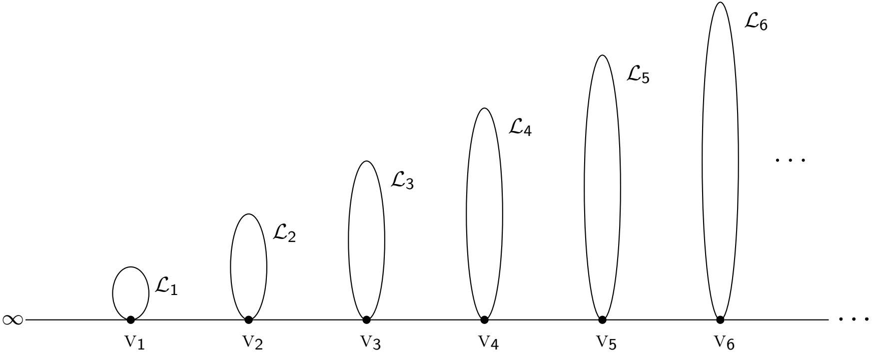
Overviews of the subject

- Adami R., Serra E., Tilli P. *Nonlinear dynamics on branched structures and networks* https://arxiv.org/abs/1705.00529 (2017)
- Kairzhan A., Noja D., Pelinovsky D. *Standing waves on quantum graphs* J. Phys. A: Math. Theor. 55 243001 (2022)

Videos

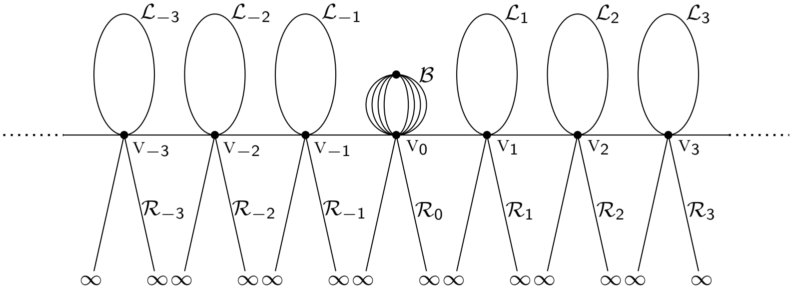
- Adami R. *Ground states of the Nonlinear Schrodinger Equation on Graphs: an overview (Lisbon WADE)* https://www.youtube.com/watch?v=G-FcnRVvoos (2020)
- Carl Wieman *Nobel Lecture* https://www.nobelprize.org/prizes/physics/2001/wieman/lecture/ (2001)
- Eric Cornell *Nobel Lecture* https://www.nobelprize.org/prizes/physics/2001/cornell/lecture (2001)
- Wolfgang Ketterle *Nobel Lecture* https://www.nobelprize.org/prizes/physics/2001/ketterle/lecture/ (2001)

Case B1



$\mathcal{L}_1$
$\mathcal{L}_2$
$\mathcal{L}_3$
$\mathcal{L}_4$
$\mathcal{L}_5$
$\mathcal{L}_6$
...
∞
...
v_1
v_2
v_3
v_4
v_5
v_6

Case B2


$\mathcal{L}_{-3}$
$\mathcal{L}_{-2}$
$\mathcal{L}_{-1}$
$\mathcal{B}$
$\mathcal{L}_1$
$\mathcal{L}_2$
$\mathcal{L}_3$
v_{-3}
v_{-2}
v_{-1}
v_0
v_1
v_2
v_3
$\mathcal{R}_{-3}$
$\mathcal{R}_{-2}$
$\mathcal{R}_{-1}$
$\mathcal{R}_0$
$\mathcal{R}_1$
$\mathcal{R}_2$
$\mathcal{R}_3$
∞